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Three-dimensional disturbances to a mixing layer in the nonlinear critical-layer regime

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We consider the nonlinear spatial evolution in the streamwise direction of slightly three-dimensional disturbances in the form of oblique travelling waves (with spanwise wavenumber k_z much less than the streamwise one k_x) in a mixing layer $v_x = u(y)$ at large Reynolds numbers. A study is made of the transition (with the growth of amplitude) to the regime of a nonlinear critical layer (CL) from regimes of a viscous CL and an unsteady CL, which we have investigated earlier (Churilov & Shukhman 1994). We have found a new type of transition to the nonlinear CL regime that has no analogy in the two-dimensional case, namely the transition from a stage of 'explosive' development. A nonlinear evolution equation is obtained which describes the development of disturbances in a regime of a quasi-steady nonlinear CL. We show that unlike the two-dimensional case there are two stages of disturbance growth after transition. In the first stage (immediately after transition) the amplitude A increases as x. Later, at the second stage, the 'classical' law $A \sim x^{2/3}$ is reached, which is usual for two-dimensional disturbances. It is demonstrated that with the growth of k_z the region of three-dimensional behaviour is expanded, in particular the amplitude threshold of transition to the nonlinear CL regime from a stage of 'explosive' development rises and therefore in the 'strongly three-dimensional' limit $k_z = O(k_x)$ such a transition cannot be realized in the framework of weakly nonlinear theory.

1. Introduction

It is well known that in linear theory the most unstable disturbances in free shear flows of homogeneous incompressible fluid are two-dimensional. Recent studies (Goldstein & Choi 1989; Wu 1993*a*; Wu, Lee & Cowley 1993; Churilov & Shukhman 1994; Goldstein 1994) of the weakly nonlinear evolution of perturbations in such flows at high Reynolds numbers have demonstrated that three-dimensional perturbations can grow explosively, i.e. much faster than two-dimensional ones.

Such a difference is not very surprising from the viewpoint of general theory of self-interacting unstable wave dynamics in shear flows (Churilov & Shukhman 1992). (In this paper we do not consider interaction between different waves, see e.g. Goldstein 1994). In the weakly nonlinear theory of high-Reynolds-number shear flows it is the critical layer (i.e. the thin layer containing the critical level at which the flow velocity is equal to the phase velocity of the wave) in which the main nonlinear processes take place. Hence the evolution behaviour of a perturbation strongly depends on the regime of the critical layer (CL) and the properties of the neutral mode in the CL. Recall from Churilov & Shukhman (1992) that the CL is viscous, unsteady or nonlinear depending on which of the three scales (viscous $l_\nu = \nu^{1/3}$, unsteady $l_t = |A^{-1}\, \mathrm{d}A/\mathrm{d}\xi|$ or nonlinear $l_N = A^{1/2}$) is greater (ν is the inverse Reynolds number, A is the wave amplitude and ξ

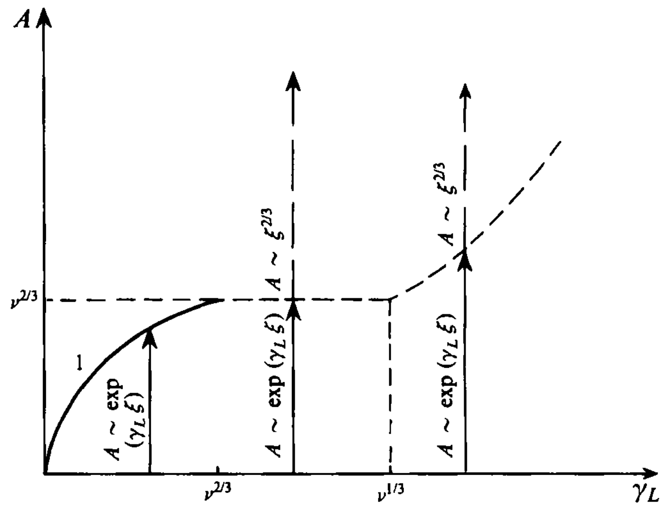

FIGURE 1. The evolution of two-dimensional disturbances. Curve 1, nonlinearity threshold in the viscous CL regime.

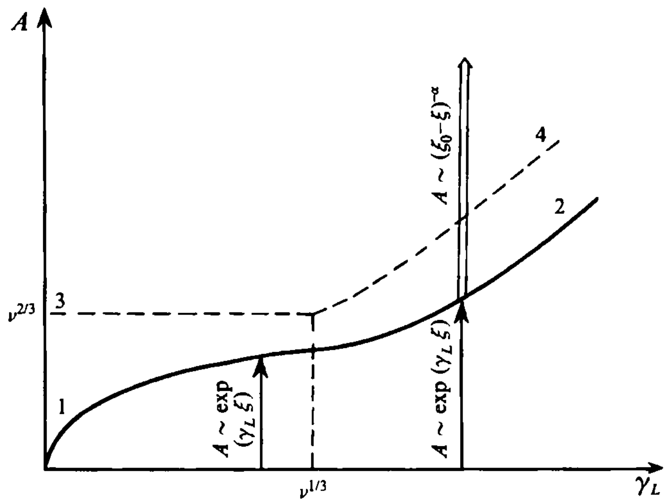

FIGURE 2. The evolution of 'strongly three-dimensional' disturbances. Curves 1 and 2, nonlinearity thresholds in the regimes of viscous and unsteady CL, respectively; curve 3, $A \sim \nu^{2/3}$, formal boundary between viscous and nonlinear CL; curve 4, $A \sim \gamma_L^2$, formal boundary between unsteady and nonlinear CL.

is the evolution variable – time or streamwise coordinate; all quantities are scaled by the characteristic velocity and width of flow).

In homogeneous incompressible flows the two-dimensional neutral mode with one CL is regular and the evolution of unstable two-dimensional perturbations is as shown in figure 1 (see Churilov & Shukhman 1992 and references therein; γ_L is the linear growth rate considered as a measure of supercriticality). When supercriticality is low enough ($\gamma_L < \nu$ or, in some flows, $\gamma_L < \nu^{2/3}$) an unstable perturbation development is described by a Landau-type evolution equation

$$\frac{dA}{d\xi} = \gamma_L A + \sum_{n=1}^{\infty} b_n A^{2n+1} \tag{1.1}$$

and instability is saturated in the viscous CL regime. (Usually, it is sufficient to take into account only the cubic nonlinearity, but in some cases the term $\sim A^5$ is also needed.) At higher supercriticality a perturbation increases exponentially to certain level ($A = O(\nu^{2/3})$ in the viscous CL regime or $A = O(\gamma_L^2)$ in the unsteady CL regime) and then transition to the nonlinear CL regime occurs. This transition is accompanied by a restructuring of the CL and results in a reduction of the growth rate (Huerre & Scott 1980; Churilov & Shukhman 1987*a*)

$$\gamma_L \to \left[-\frac{1}{\pi}\Phi\left(\frac{\nu}{A^{3/2}}\right)\right]\gamma_L; \quad \Phi(z) = \begin{cases} Cz + O(z^2), & z \ll 1 (C<0) \\ -\pi + O(z^{-4/3}), & z \gg 1, \end{cases} \tag{1.2}$$

which leads to a change in the exponential growth of amplitude for a slow power-law growth as $A \sim \xi^{2/3}$ in accordance with the nonlinear evolution equation (NEE)

$$\frac{\mathrm{d}A}{\mathrm{d}\xi} = -\frac{\Phi}{\pi}\gamma_L A \approx -\frac{C\nu}{\pi}\gamma_L A^{-1/2}. \tag{1.3}$$

Note that in both (1.1) and (1.3) the nonlinearity is an algebraic (local) one, i.e. it is determined by the value of the amplitude at a current point ξ.

Three-dimensional (oblique) neutral modes are singular (at least one physical quantity has a singularity), and when the applicability conditions of Squire's theorem are not satisfied, as discussed in detail by Churilov & Shukhman (1994, hereinafter referred to as C & S), the unstable perturbation evolution closely resembles the behaviour of two-dimensional disturbances in those flows where the neutral mode on the CL is singular as, for example, in stratified (Churilov & Shukhman 1987*b*, 1988) or compressible (Goldstein & Leib 1989; Shukhman 1991) flows.

A corresponding type of evolution behaviour is shown in figure 2. Here the threshold of nonlinearity (the amplitude at which the nonlinearity becomes competitive), both in the regime of a viscous CL ($\gamma_L < \nu^{1/3}$, curve 1) and in the regime of an unsteady CL ($\gamma_L > \nu^{1/3}$, curve 2), is below the formal boundary of a nonlinear CL (lines 3 and 4); therefore, at any supercriticality γ_L the evolution is already becoming nonlinear (i.e. nonlinearity begins to play an important role in the dynamics of disturbances) in linear (i.e. viscous or unsteady) CL regimes.

In the regime of a viscous CL, upon reaching the threshold of nonlinearity, the amplitude growth stops (C & S) or even reverses (Wu *et al.* 1993) leading to the total dissipation of the disturbance. In the regime of an unsteady CL, however, the intersection of the nonlinearity threshold leads to an explosive growth of the disturbance as $A \sim (\xi_0 - \xi)^{-3}$ for a pair of oblique waves (Goldstein & Choi 1989; Wu *et al.* 1993), or $A \sim (\xi_0 - \xi)^{-5/2}$ for a single travelling wave (C & S). During this explosive growth the unsteady scale l_t remains larger than the nonlinear one l_N up to $A = O(1)$, so that there is no evolution to the regime of a nonlinear CL.

To describe 'explosive' processes requires a totally different (from (1.2) and (1.3)) type of NEE, the NEE with non-local nonlinearity of the form

$$\frac{\mathrm{d}A}{\mathrm{d}\xi} = \gamma_L A - b\int_0^\infty \mathrm{d}\zeta\,\zeta^p \int_0^1 \mathrm{d}\sigma\, K(\sigma)\, A(\xi-\zeta)\, A(\xi-\sigma\zeta)\, \bar{A}(\xi-(1+\sigma)\zeta) \tag{1.4}$$

first obtained by Hickernell (1984). The explicit form of the kernel $K(\sigma) = O(1)$ is specific to each problem; the overbar denotes complex conjugacy. Similar equations have been obtained for stratified (Churilov & Shukhman 1988) and compressible (Goldstein & Leib 1989; Shukhman 1991) flows in the two-dimensional case. Note also that in the case of a three-dimensional disturbance the NEE has non-local nonlinearity

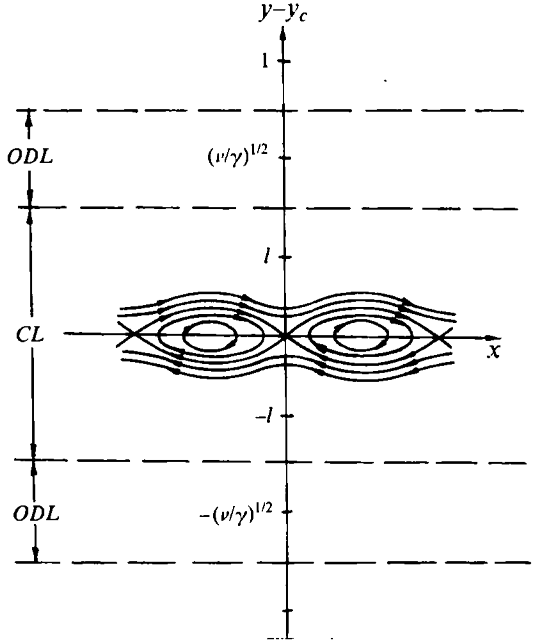

FIGURE 3. Structure of the inner region. CL, critical layer: $\epsilon^{1/2} \sim |y-y_0| \ll (\epsilon\eta/\mu)^{1/2}$; ODL, outer diffusion layers: $(\epsilon\eta/\mu)^{1/2} \sim |y-y_0| \ll 1$.

even in the regime of a viscous CL (Wu *et al.* 1993; Wu 1993*b*; Smith & Blennerhassett 1992; C & S) and is

$$\frac{\mathrm{d}|A|^2}{\mathrm{d}\xi} = 2\gamma_L|A|^2 + b|A|^2 \int_0^\infty \mathrm{d}\zeta\, \zeta^q |A(\xi-\zeta)|^2. \tag{1.5}$$

These are the main differences in the evolutionary behaviour of three- and two-dimensional disturbances.

One can legitimately ask what is the 'degree of three-dimensionality' at which the 'two-dimensional' behaviour changes to a 'three-dimensional' one. The answer can be found by studying the changes when k_z is varied from 0 to $O(k_x)$ (**k** is a wave vector of the disturbance, the x-axis is streamwise, and the y-axis is along the velocity gradient). In this paper we try to do this.

Note that the coefficient b in equations (1.4) and (1.5) is proportional to k_z^2. When $k_z^2 \ll 1$, 'oblique' terms in the NEE are reduced and are only able to withstand competition from traditional ('two-dimensional') nonlinearity in some part of a 'weakly-nonlinear domain' (defined by unequalities $|A| \ll 1$, $\gamma_L \ll 1$), but in the other part of this domain the perturbation behaviour becomes two-dimensional: a growth of amplitude would lead (at a sufficiently large supercriticality $\gamma_L > \nu^{2/3}$) to the transition to the regime of a nonlinear CL. According to estimates made in C & S, this takes place at $A \sim \nu^{2/3}$ if the perturbation growth starts in the viscous CL regime ($\gamma_L < \nu^{1/3}$) and at $A \sim \max(k_z^2, \gamma_L^2)$ if it starts in the unsteady CL regime ($\gamma_L > \nu^{1/3}$).

Now let us recall the main point of transition to the nonlinear CL regime and the further evolution of this regime of two-dimensional perturbations. It is well known that in perturbed shear flow the picture of streamlines is of Kelvin's cat's eyes (figure 3) inside which fluid particles are trapped. From the physical point of view the transition to the nonlinear CL regime means that the characteristic time $t_N = l_N^{-1}$ of trapped particle motion in the wave becomes shorter than both the time of vorticity diffusion $t_\nu = l^2/\nu$ (on the CL scale l) and the characteristic time of evolution (i.e. time of capture of new particles) $t_e = l_t^{-1}$. In other words, this transition takes place when trapped particles have enough time for mixing inside the cat's eyes. (In spatial-evolution problems we replace 'time' by 'streamwise distance'.)

In the inviscid limit ($\nu = 0$) the vorticity is an integral of the motion and each fluid particle carries its own value of vorticity along its trajectory. Because the orbits of trapped particles are non-isochronous, the particle mixing leads to a highly oscillating fine-scale distribution of vorticity inside the cat's eyes. As a result an exponential growth of amplitude changes to damping oscillations around some constant value (Stewartson 1978; Churilov & Shukhman 1987*a*; see also Goldstein & Leib 1988). It should be emphasized that two-dimensional perturbations move into the nonlinear CL regime from a *linear stage of development* (i.e. from the stage of exponential growth). The presence of viscosity leads, first of all, to damping of the fine-scale structure of the vorticity distribution due to particle mixing and to the formation of a smooth 'plateau' in the vorticity profile inside the cat's eyes (Churilov & Shukhman 1987*a*). Also, viscosity tries to maintain the amplitude growth but much slower ($A \sim \xi^{2/3}$) than before transition (Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988). Note that the character of the transient behaviour strongly depends on the value of ν: at high viscosity (low supercriticality, $\gamma_L < \nu^{1/3}$) the wave amplitude varies monotonically and the process of transition as a whole is quasi-steady, but at low viscosity (high supercriticality, $\gamma_L > \nu^{1/3}$) amplitude growth becomes monotonic only after some oscillations, and the transient CL dynamics is highly unsteady (see figure 4 in Churilov & Shukhman 1987*a*). After transition to the nonlinear CL regime the perturbation evolution (described by NEE (1.3)) is quasi-steady and qualitatively the same ($A \sim \xi^{2/3}$) over wide range of ν. In other words, from both CL regimes, viscous and unsteady, a perturbation moves into *the same* nonlinear CL regime.

In this paper we shall show that weakly three-dimensional disturbances ($k_z^2 \ll 1$) with a growth of amplitude also move into the regime of a nonlinear CL, but this transition differs greatly from the two-dimensional one when k_z^2 is not too small ($k_z^2 \gg \nu^{2/3}$). Namely, perturbations move into the nonlinear CL regime from a *stage of nonlinear development* at $\nu^{2/3} < \gamma_L < k_z^2$ and from a stage of linear development (as in the two-dimensional case) at higher supercriticality. In the most interesting case ($k_z^2 \gg \nu^{1/3}$) transition from the viscous CL regime ($\nu^{2/3} < \gamma_L < \nu^{1/3}$) is preceded by a phase of power-law growth $A \sim \xi^{1/4}$ in accordance with equation (1.5), and transition from the unsteady CL regime ($\nu^{1/3} < \gamma_L < k_z^2$) is preceded by an explosive growth $A \sim (\xi_0 - \xi)^{-5/2}$ governed by equation (1.4). It should be noted that the transition to the regime of a nonlinear CL through the explosive growth phase in the regime of an unsteady CL is a totally new type of evolution, specific precisely to weakly three-dimensional disturbances: in a strongly three-dimensional case ($k_z = O(1)$, C&S) or in two-dimensional problems with singular neutral modes (Churilov & Shukhman 1992) the amplitude at which such a transition occurs cannot be attained in the framework of weakly nonlinear theory.

Because the transition to the regime of a nonlinear CL takes place from a *nonlinear* stage of perturbation development it is accompanied not only by a reduction (1.2) of growth rate but also by a reduction of nonlinearity in the NEE. We have calculated such a reduction for quasi-steady transition from the viscous CL regime and find that for the nonlinear term of the NEE (1.5) the reduction factor is even stronger than for the linear one and is equal to $(-\Phi/\pi)^2$. In the formation of this new disturbance dynamics quite a non-trivial role is played by the outer diffusion layers (ODL, see figure 3) sandwiching a quasi-steady (i.e. viscous or nonlinear) CL and connecting it with the almost non-dissipative linear and unsteady regions of the external flow (one should distinguish the ODLs from (inner) diffusion layers situated near boundaries of cat's eyes, e.g. Haberman 1972; Brown & Stewartson 1978). (Note that non-local nonlinearity in (1.5) is also due to ODLs (Wu 1993*b*; C&S).)

Transition from the unsteady CL regime clearly includes the stage of oscillatory amplitude development due to trapped particle mixing, but after the formation and viscous damping of a fine-scale highly oscillating vorticity distribution inside the cat's eyes the amplitude evolution becomes quasi-steady and monotonic, and further development proceeds *in the same* nonlinear CL regime as after transition from a viscous CL. To study the transition from an unsteady CL regime in detail the numerical solution of appropriate equations ((2.13)–(2.15)) is needed and this will be done in a separate work. However, the physical nature of the transient processes is quite transparent and we have no reason to doubt that picture of evolution described above is correct.

The purpose of this paper is to study – in the framework of the simplest model of a plane shear flow – the transition from two-dimensional to three-dimensional behaviour as k_z increases. We investigate the spatial evolution of the travelling wave (the temporal one is 'two-dimensional' at arbitrary k_z because the applicability conditions for Squire's theorem are valid; for further detail see C&S). Namely, we consider the downstream (x-direction) development of an oblique (travelling at some small angle to the stream) wave generated by a source of frequency ω located near the origin of the mixing layer (formally, at $\xi = -\infty$). This problem may seem somewhat artificial because it is very difficult to excite *a single* weakly oblique plane wave. However, the plane mixing layer is a reasonably good approximation for, say, an axial jet with small curvature (i.e. ratio of shear layer thickness to its radius) in which a single low-m travelling helical wave can be generated in an experiment (see C&S and references therein). On the other hand, the problem considered in our paper is rather complicated and the choice of as simple a flow model as possible is appropriate.

The paper is organized as follows: in §2 we formulate the problem and derive the equations describing the perturbation behaviour inside the CL; the structure of the quasi-steady nonlinear CL is defined and NEE are derived in §3; and §4 is concerned with a discussion of the results obtained. The Appendix gives some useful properties of the solutions of the well-known nonlinearly viscous equation arising in the case of a nonlinear steady-state CL.

2. Problem statement and basic equations

Consider a plane shear flow $v_x = u(y)$ (for simplicity, it is assumed that $u(y)$ is a monotonically increasing function). We write the Navier–Stokes equations as

$$\left.\begin{aligned} \nabla\cdot\boldsymbol{v} &= 0,\\ \frac{\partial \boldsymbol{v}}{\partial t}+u(y)\frac{\partial \boldsymbol{v}}{\partial x}+u'v_y\boldsymbol{e}_x+(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} &= -\nabla p+\nu\nabla^2\boldsymbol{v},\end{aligned}\right\} \tag{2.1}$$

where $\boldsymbol{v}$ and p are, respectively, the velocity and pressure perturbations, $\boldsymbol{e}_x$ is the unit vector in the x-direction. All quantities are scaled by a characteristic flow velocity and shear layer width, and a prime denotes the y-derivative. Let a disturbance be produced by an external source with frequency ω and let it propagate in the form of a wave travelling at an angle to the flow with a (two-dimensional) wave vector $\boldsymbol{k} = (k_x, k_z)$ (in what follows everywhere $k_x \equiv k$), but let it evolve strictly downstream, i.e. in x. The construction of the outer solution faithfully copies that reported in C&S; one needs only to put $r \to \infty$, $r_c/r = 1$, and $m/r_c = k_z$. As the main perturbation, as in C & S, we choose a pressure perturbation in a linear approximation which is conveniently expressed immediately in a real form

$$p = 2\epsilon B g(y) \cos(kx + k_z z - \omega t + \Theta),$$

where $\epsilon \ll 1$ is a parameter characterizing the disturbance amplitude, and $g(y)$ is an eigenfunction of the neutral mode with $k_z = 0$ calibrated by the condition $g = 1$ at a critical level, i.e. when $y = y_c$; amplitude $A = \epsilon B$ and phase Θ are slowly evolving downstream (in x). It is assumed that the disturbance is a weakly supercritical one, i.e. the Strouhal number is slightly less than a critical value:

$$\omega_{cr} - \omega \ll \omega_{cr}, \quad \text{where} \quad \omega_{cr} = \omega_0 - ck_z^2/2k;$$

ω_0 is the frequency (Strouhal number) of the neutral mode with $k_z = 0$, k is a corresponding wavenumber, and $c = \omega_0/k$.

In a CL we need to be able to take into account the viscosity, the 'non-stationarity' (we have conserved this term, adopted in temporal evolution problems, also for the spatial evolution case) and the nonlinearity simultaneously; therefore, the three scales (see the Introduction)

$$l_\nu \sim \nu^{1/3}, \quad l_t \sim |B^{-1}\,\mathrm{d}B/\mathrm{d}x|, \quad l_N \sim \epsilon^{1/2}$$

are assumed to be of the same order of magnitude. In addition, because the transition to the regime of a nonlinear CL from the regime of an unsteady CL occurs when $A \sim k_z^4$ (see C & S), it is necessary to assume $k_z = O(\epsilon^{1/4})$. Accordingly, we introduce

$$\left.\begin{aligned} &\xi = \mu\epsilon^{1/2}x, \quad \zeta = \epsilon^{1/4}z, \quad y - y_c = \epsilon^{1/2}Y, \\ &\nu = \epsilon^{3/2}\eta, \quad \omega = \omega_0 + \mu\epsilon^{1/2}\Omega, \quad k_z = \epsilon^{1/4}q. \end{aligned}\right\} \tag{2.2}$$

The parameters η and μ are, generally speaking, of $O(1)$. However, when it is necessary to consider a particular regime of a CL, their order should be varied. Thus, the regime of a viscous CL corresponds to $\eta \gg 1$, the regime of an unsteady CL to $\mu \gg 1$, and the regime of a nonlinear CL to $\eta \ll 1$, $\mu \ll 1$.

Strictly speaking, the parameter Ω should be introduced not as the measure of the deviation of the frequency of the excited disturbance from the frequency of the neutral mode with $k_z = 0$ but as the measure of supercriticality, i.e. by the relationship $\omega = \omega_{cr} + \mu\epsilon^{1/2}\Omega$, where $\omega_{cr} = \omega_0 - cq^2\epsilon^{1/2}/2k$. However, such an Ω would only increase the unwieldiness of the formulae, without changing the final form of the NEE (see, for example, (3.28) and (4.4a)).

The inner ($y \to y_c$) asymptotic representation of the outer solution has the form (only the pertinent terms are written out, $\boldsymbol{v} = \{u, v, w\}$)

$$p = 2\epsilon B_c + \epsilon^2\left[\mu\frac{2k}{u_c'}(\Omega B_c - c\dot{B}_s)\,Y - k^2 B_c\,Y^2 - \frac{2k^2}{u_c'^2}B^2\right] + \ldots, \tag{2.3a}$$

$$\begin{aligned} v = {} & \epsilon\frac{2k}{u_c'}B_s + \epsilon^{3/2}\left\{\frac{6\alpha}{ku_c'}\left[\frac{\mu}{ku_c'}(\Omega B_s + c\dot{B}_c) - B_s\,Y\right] - \frac{2\mu}{u_c'}\dot{B}_c + \frac{2q^2}{ku_c'}B_s\right\} \\ & + \epsilon^2\left\{\mu\left[\frac{2q^2}{k^2u_c'}\dot{B}_c + \frac{2u_c'''}{u_c'^3}(\Omega B_s + c\dot{B}_c)\,Y\ln|\epsilon^{1/2}Y|\right] + d_1^{\pm}\,Y\right. \\ & \left. + \frac{kB_s\,Y^2}{u_c'}\left(\frac{u_c'''}{u_c'} + k^2\right)\right\} + \ldots, \end{aligned} \tag{2.3b}$$

$$w = \epsilon^{3/4}\left(-\frac{2qB_c}{ku_c'\,Y} + \frac{2qB^2}{ku_c'^3\,Y^3} + \ldots\right), \tag{2.3c}$$

where, for brevity, we introduce the designations

$$B_c = B\cos\vartheta, \quad B_s = B\sin\vartheta; \quad \vartheta = kx + q\zeta - \omega t + \Theta(\xi), \quad \dot{B}_{c,s} = \frac{\partial}{\partial\xi}(B_{c,s}),$$

and the prime denotes derivative with respect to y, $u'_c = u'(y_c)$.

The coefficient α (in (2.3*b*)) of the expansion of $g(y)$ in the vicinity of $y = y_c$,

$$g = 1 - \tfrac{1}{2}k^2(y-y_c)^2 + \alpha(y-y_c)^3 + \dots,$$

is determined only as a result of solving the boundary-value problem for g,

$$\frac{d^2g}{dy^2} - \frac{2u'}{u-c}\frac{dg}{dy} - k^2 g = 0, \quad g \to 0 \quad \text{as} \quad y \to \pm\infty,$$

and we will not need it in an explicit form.

The solution of the outer problem is also known to give a relationship between the coefficients d_1^+ and d_1^- (see (2.3*b*)), the so-called modified solvability condition (MSC):

$$d_1^+ - d_1^- = 4\frac{\mu u'_c}{k^2}(\Omega B_s + c\dot{B}_c) I_1 - 2\frac{u'_c}{k}(2\mu k \dot{B}_c - q^2 B_s) I_2, \tag{2.4}$$

where
$$I_1 = \int_{-\infty}^{\infty} dy \frac{u'g'g}{(u-c)^4} + \frac{i\pi k^2 u'''_c}{2u'^4_c}, \quad I_2 = -\frac{1}{k^2}\int_{-\infty}^{\infty} dy \frac{g'^2}{(u-c)^2} < 0, \tag{2.5}$$

and the singular point $y = y_c$ is indented from below (because $u'_c > 0$). Note that in the expression for I_1 the second term on the right-hand side represents a semi-residue (with an opposite sign) in $y = y_c$, hence in general I_1 is an integral in the sense of the principal value. In a frequently used model with the velocity profile $u(y) = c + \tanh y$ we have: $y_c = 0$, $u'_c = 1$, $u'''_c = -2$, $k = 1$, $\omega_0 = c$, $g(y) = 1/\cosh y$, $I_1 = 0$, and $I_2 = -2$.

When solving the inner problem, one gets a relationship of the form (2.4), but with a different right-hand side. By comparing it with (2.4), one obtains nonlinear evolution equations for $B(\xi)$ and $\Theta(\xi)$.

For constructing the solution inside a CL, we introduce

$$u = \epsilon U, \quad v = \epsilon V, \quad w = \epsilon^{3/4} W, \quad p = \epsilon P$$

and pass to the variables Y, ζ and ξ. Equations (2.1) take the form

$$V_Y + \epsilon^{1/2}(U_x + W_\zeta) + \epsilon\mu U_\xi = 0, \tag{2.6a}$$

$$(u'_c Y - \mu\Omega/k) U_x + \mu c U_\xi + V U_Y - \eta U_{YY} + \mu P_\xi = -\epsilon^{-1/2}(P_x + u'_c V)$$
$$- \epsilon^{1/2}(\mu u'_c Y U_\xi + U U_x + W U_\zeta + \tfrac{1}{2}u'''_c Y^2 V) + O(\epsilon), \tag{2.6b}$$

$$(u'_c Y - \mu\Omega/k) V_x + \mu c V_\xi + V V_Y - \eta V_{YY} = -\epsilon^{-1} P_Y + O(\epsilon^{1/2}), \tag{2.6c}$$

$$(u'_c Y - \mu\Omega/k) W_x + \mu c W_\xi + V W_Y - \eta W_{YY} + P_\zeta$$
$$= -\epsilon^{1/2}(\mu u'_c Y W_\xi + U W_x + W W_\zeta) + O(\epsilon). \tag{2.6d}$$

We search for a solution of this system in the form of an expansion

$$f = f^{(1)} + \epsilon^{1/2} f^{(2)} + \epsilon f^{(3)} + \dots.$$

To leading order equations (2.6*a*–*c*) yield

$$V_Y^{(1)} = 0, \quad P_x^{(1)} + u'_c V^{(1)} = 0, \quad P_Y^{(1)} = 0,$$

which after matching to (2.3 a, b) leads to

$$P^{(1)} = 2B_c, \quad V^{(1)} = \frac{2k}{u_c'} B_s. \tag{2.7}$$

From equation (2.6 d) using (2.7) we obtain

$$\mathscr{L} W^{(1)} = 2qB_s, \tag{2.8}$$

where
$$\mathscr{L} = (u_c' Y - \mu\Omega/k)\frac{\partial}{\partial x} + \mu c \frac{\partial}{\partial \xi} - \eta \frac{\partial^2}{\partial Y^2} + \frac{2k}{u_c'} B_s \frac{\partial}{\partial Y}. \tag{2.9}$$

The second iteration of equation (2.6 c) gives $P_Y^{(2)} = 0$, and matching to (2.3 a) shows that $P^{(2)} = 0$. Next, from the system (2.6) we obtain

$$V_Y^{(2)} + U_x^{(1)} + W_\xi^{(1)} = 0, \tag{2.10a}$$

$$\mathscr{L} U^{(1)} = -u_c' V^{(2)} - \mu P_\xi^{(1)} \equiv -u_c' V^{(2)} - 2\mu \dot{B}_c, \tag{2.10b}$$

$$\mathscr{L} V^{(1)} \equiv \frac{2k}{u_c'}(ku_c' Y - \mu\Omega) B_c + \frac{2\mu ck}{u_c'} \dot{B}_s = -P_Y^{(3)}, \tag{2.10c}$$

$$\mathscr{L} W^{(2)} = -V^{(2)} W_Y^{(1)} - (\mu u_c' Y W_\xi^{(1)} + U^{(1)} W_x^{(1)} + W^{(1)} W_\zeta^{(1)}). \tag{2.10d}$$

In view of the fact that $\partial/\partial x = (k/q)\partial/\partial\zeta$, equation (2.10 a) is integrated to give

$$\tilde{G}_Y + U^{(1)} + \frac{q}{k} W^{(1)} = 0, \quad \tilde{G}_x = V^{(2)}.$$

By acting with the operator $(\partial/\partial Y)\mathscr{L}$ on the result and using (2.7) and (2.8), we obtain $\mathscr{L}\tilde{G}_{YY} = 0$, from which $V_{YY}^{(2)} = 0$ follows. Matching to (2.3 b) gives

$$V^{(2)} = \frac{6\alpha}{ku_c'}\left[\frac{\mu}{ku_c'}(\Omega B_s + c\dot{B}_c) - B_s Y\right] - \frac{2\mu}{u_c'} \dot{B}_c + \frac{2q^2}{ku_c'} B_s.$$

Finally, from (2.10 a) and (2.10 c) we obtain

$$U^{(1)} = -\frac{6\alpha}{k^2 u_c'} B_c - \frac{q}{k} W^{(1)},$$

$$P^{(3)} = -k^2 B_c Y^2 + \frac{2k}{u_c'} \mu(\Omega B_c - c\dot{B}_s) Y - \frac{2k^2}{u_c'^2} B^2.$$

In a third iteration we need only the first two equations of the system (2.6):

$$V_Y^{(3)} + U_x^{(2)} + W_\xi^{(2)} + \mu U_\xi^{(1)} = 0, \tag{2.11a}$$

$$\mathscr{L} U^{(2)} = -P_x^{(3)} - u_c' V^{(3)} - V^{(2)} U_Y^{(1)}$$
$$- (\mu u_c' Y U_\xi^{(1)} + U^{(1)} U_x^{(1)} + W^{(1)} U_\zeta^{(1)} + \tfrac{1}{2} u_c''' Y^2 V^{(1)}). \tag{2.11b}$$

By integrating (2.11 a) over x,

$$G_Y^* + U^{(2)} + \frac{q}{k} W^{(2)} + \mu U_\xi^* = 0,$$

where
$$G_x^* = V^{(3)}, \quad U_x^* = U^{(1)},$$

and acting on the result with the operator $(\partial/\partial Y)\mathscr{L}$ we obtain

$$\mathscr{L}G^*_{YY} = -\mu\frac{\partial}{\partial Y}\mathscr{L}U^*_\xi - \frac{6\alpha}{k^2}\mu\dot{B}_c + 2k(u'''_c/u'_c+k^2)\,B_s\,Y - \frac{2k^2}{u'_c}\mu(\Omega B_s + c\dot{B}_c).$$

The asymptotic expansion (2.3b) contains at $O(\epsilon^2)$ a term $\sim Y^2$. We separate from G^* the respective contribution by putting

$$G^* = G - \frac{1}{u'_c}(u'''_c/u'_c+k^2)\,B_c\,Y^2.$$

Then
$$\mathscr{L}\left(G_{YY} - \mu\frac{q}{k}W^*_{Y\xi}\right) = \mu\frac{q}{k}u'_c\,W^{(1)}_\xi + \frac{2u'''_c}{u'^2_c}\mu(\Omega B_s + c\dot{B}_c), \tag{2.12}$$

where $W^*_x = W^{(1)}$.

Equations (2.8) and (2.12) are the basic equations that define the solution inside the CL and the CL contribution to the MSC (2.4). It is convenient to drop the upper index and write them as

$$\mathscr{L}W = 2qB_s, \tag{2.13a}$$

$$\mathscr{L}F = \mu\frac{q}{k}u'_c\,W_\xi, \tag{2.13b}$$

$$\mathscr{L}T = \frac{2u'''_c}{u'^2_c}\mu(\Omega B_s + c\dot{B}_c), \tag{2.13c}$$

$$G_{YY} = F + T + \mu\frac{q}{k}W^*_{Y\xi}, \tag{2.13d}$$

which clearly demonstrates that T defines a 'two-dimensional' contribution to the vorticity, and W and F represent a 'three-dimensional' contribution. Matching to (2.3b, c) requires quite a definite behaviour as $Y \to \pm\infty$:

$$W \sim -\frac{2qB_c}{ku'_c\,Y} + \frac{2qB^2}{ku'^3_c\,Y^3} + \dots, \tag{2.14a}$$

$$F \sim -\frac{2\mu q^2\dot{B}_s}{k^3u'_c\,Y^2} + \dots, \tag{2.14b}$$

$$T \sim -\frac{2\mu u'''_c}{ku'^3_c}(\Omega B_c - c\dot{B}_s)\,Y^{-1} + \dots. \tag{2.14c}$$

The MSC (2.4) is conveniently written as

$$\frac{2u'_c}{k^3}\left[I_0\left(\mu\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} + \frac{q^2}{2k}\right) - I_1(\mu\Omega + cq^2/2k)\right]B = \int_{-\infty}^{\infty}\mathrm{d}Y\langle G_{YY}\cos\vartheta\rangle, \tag{2.15a}$$

$$\frac{2\mu u'_c}{k^3}I_0\frac{\mathrm{d}B}{\mathrm{d}\xi} = \int_{-\infty}^{\infty}\mathrm{d}Y\langle G_{YY}\sin\vartheta\rangle, \tag{2.15b}$$

where
$$\int_{-\infty}^{\infty}\mathrm{d}Y(\dots) = \lim_{z\to\infty}\int_{-z}^{z}\mathrm{d}Y(\dots);\quad \langle\dots\rangle = \frac{1}{2\pi}\int_0^{2\pi}\mathrm{d}\vartheta(\dots),$$

$$I_0 \equiv cI_1 - k^2I_2.$$

Equations (2.13)–(2.15) describe the evolution of disturbances in the three CL regimes: viscous, unsteady and nonlinear. One may demonstrate that in the limiting cases of an unsteady and viscous CL the nonlinear evolution equations that follow from (2.13)–(2.15) coincide, respectively, with equations (4.1) and (4.5) from C & S (in the limit of a plane flow). The limit of a nonlinear CL will be considered below.

3. Quasi-steady nonlinear CL

The goal of this Section is to calculate the integrals on the right-hand sides of (2.15*a*, *b*). Each of them contains three contributions. It is easy to see that by virtue of (2.14*a*) the contribution due to W^* is zero, hence there actually are only two contributions: 'two-dimensional' (caused by T) and 'three-dimensional' (caused by F). In the linear (viscous and unsteady) CL regimes they have already been calculated by C & S; therefore, here we are interested mainly in the regime of a nonlinear CL and in the transition to it.

As was discussed in the Introduction, experience of solving similar two-dimensional problems (Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988) shows that the evolution leads to a quasi-steady nonlinear CL. Indeed, the evolution that starts in the viscous CL regime ($\gamma_L < \nu^{1/3}$) continues to proceed in a quasi-steady manner ($\mu \ll \min(\eta, 1)$). In the case of the transition from the regime of an unsteady CL ($\gamma_L > \nu^{1/3}$), after a sufficiently fast relaxation stage, a smooth distribution is reached on the vorticity profile inside the cat's eyes, the evolution rate is decelerated so that the relationship

$$\mu \ll \eta \ll 1 \tag{3.1}$$

is established, and main unsteady processes are pushed to the region $1 \ll |Y| \ll \epsilon^{-1/2}$. Thus the region, which to this point we have called the CL, is divided into three: a central layer (as before, we will call it the CL because it is in this region where the resonant wave–particle interaction takes place); and two outer diffusion layers (ODL) with a scale $|Y| \sim \delta^{-1} = (\eta/\mu)^{1/2}$ (figure 3) sandwiching the central CL and acting like a buffer between it and the outer regions. Inside the ODLs viscous and unsteady terms have the same order.

ODLs are not new in critical-layer analysis. They are present in all problems involving quasi-steady (viscous or nonlinear) CLs and $\gamma_L > \nu$ (e.g. Churilov & Shukhman 1987*a*, *b*). In two-dimensional problems ODLs play a passive role: they are necessary only for matching a mean-flow distortion (zeroth harmonic of the perturbation) inside the CL to those in the outer regions of the flow.† In contrast, in three-dimensional problems ODLs can play an active role: they contribute to nonlinear terms in the evolution equations, e.g. the nonlinear term in (1.5) is due to such a contribution (Wu 1993*b*; Wu *et al.* 1993; C & S). Until now active ODLs have only been studied in the case of a viscous CL. In this paper we will study the nonlinear CL regime and the (active) role of ODLs in the occurrence of nonlinearity in the corresponding NEE.

We will surmise that in the weakly three-dimensional case under consideration the transition from both viscous and unsteady CL regimes finishes in the *same quasi-steady nonlinear CL regime* (see the Introduction), and we will calculate the integrals involved in the MSC (2.15*a*, *b*) in the quasi-steady limit; after that, we will check the results obtained for self-consistency. It is found that one can obtain a unified expression for the desired integrals in the entire region of quasi-stationarity, i.e. in the region of a

† Haberman (1972) was the first who pointed out that in some problems the outer asymptotic expansion of a solution inside a CL does not match to inner asymptotic expansions of solutions in the outer regions of the flow.

viscous CL and in the region of a quasi-steady nonlinear CL. For 'two-dimensional' contributions this has already been done (see, for example, C & S):

$$\oint \mathrm{d}Y\langle T\sin\vartheta\rangle = \frac{\mu u_c''' \Phi_1(\lambda)}{k u_c'^3}\left(c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi}-\Omega\right)B, \tag{3.2a}$$

$$\oint \mathrm{d}Y\langle T\cos\vartheta\rangle = -\frac{\mu u_c''' \Phi_2(\lambda)}{k u_c'^3}c\frac{\mathrm{d}B}{\mathrm{d}\xi};\quad \lambda = \eta\frac{u_c'^2}{k}(2B)^{-3/2}. \tag{3.2b}$$

The functions $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are defined, for example, in C & S; we need only to know their asymptotic behaviour:

$$\Phi_1(\lambda) = \begin{cases} C^{(1)}\lambda + O(\lambda^2), & \lambda \ll 1;\quad C^{(1)} = -5.52 \\ -\pi + O(\lambda^{-4/3}), & \lambda \gg 1, \end{cases} \tag{3.3a}$$

$$\Phi_2(\lambda) = \begin{cases} C^{(2)}\lambda^{-1} + O(1), & \lambda \ll 1;\quad C^{(2)} = -2.72 \\ -\pi + O(\lambda^{-4/3}), & \lambda \gg 1. \end{cases} \tag{3.3b}$$

Let us now calculate the 'three-dimensional' contribution. Inside the CL it is convenient to use, instead of the variables x, Y and ξ, the variables ϑ, Z and τ. Equations (2.13 *a, b*) take the form

$$\mathscr{M}w = -\frac{\mu}{k(2B)^{1/2}}\left[\frac{\partial w}{\partial\tau}+(\dot{\Theta}-\Omega)\frac{\partial w}{\partial\vartheta}+\frac{\dot{B}}{2B}\left(w-Z\frac{\partial w}{\partial Z}\right)\right], \tag{3.4}$$

$$\mathscr{M}F = -\frac{\mu}{k(2B)^{1/2}}\left[\frac{\partial F}{\partial\tau}+(\dot{\Theta}-\Omega)\frac{\partial F}{\partial\vartheta}-\frac{\dot{B}}{2B}Z\frac{\partial F}{\partial Z}\right]+\mu\frac{qu_c'}{ck^2}\left[\frac{\partial w}{\partial\tau}+\dot{\Theta}\frac{\partial w}{\partial\vartheta}+\frac{\dot{B}}{2B}\left(w-Z\frac{\partial w}{\partial Z}\right)\right], \tag{3.5}$$

where
$$\mathscr{M} = Z\frac{\partial}{\partial\vartheta}+\sin\vartheta\frac{\partial}{\partial Z}-\lambda\frac{\partial^2}{\partial Z^2}, \tag{3.6}$$

$$W = \tilde{W}+(qu_c'/k)\,Y,\quad \tilde{W} = (2B)^{1/2}w,\quad Z = u_c'\,Y/(2B)^{1/2},\quad \tau = \xi/c,$$

and the dot denotes differentiation with respect to τ.

In ODLs it is convenient to introduce

$$S = Y\delta,\quad \text{where}\quad \delta \equiv (\mu/\eta)^{1/2}$$

and to write (2.13 *a, b*) as

$$ku_c' S\frac{\partial\tilde{W}}{\partial\vartheta} = -\delta^2\frac{2k}{u_c'}B_s\frac{\partial\tilde{W}}{\partial S}+\mu\delta\left[\frac{\partial^2\tilde{W}}{\partial S^2}-\frac{\partial\tilde{W}}{\partial\tau}-(\dot{\Theta}-\Omega)\frac{\partial\tilde{W}}{\partial\vartheta}\right], \tag{3.7}$$

$$ku_c' S\frac{\partial F}{\partial\vartheta} = -\delta^2\frac{2k}{u_c'}B_s\frac{\partial F}{\partial S}+\mu\delta\left[\frac{\partial^2 F}{\partial S^2}-\frac{\partial F}{\partial\tau}-(\dot{\Theta}-\Omega)\frac{\partial F}{\partial\vartheta}+q\frac{u_c'}{ck}\left(\frac{\partial\tilde{W}}{\partial\tau}+\dot{\Theta}\frac{\partial\tilde{W}}{\partial\vartheta}\right)\right]. \tag{3.8}$$

The scheme for constructing the solution is thus: the solution in the CL is matched to the solutions in the ODLs which are then matched to (2.14 *a, b*). The solution inside the CL is sought in the form of a double expansion:

$$f = f^{(0)}+\delta f^{(1)}+\delta^2 f^{(2)}+\dots,\quad f^{(i)} = \sum_{n=0}^{\infty}\lambda^n f^{(i,n)}.$$

Solutions in the ODLs are constructed in the form of expansions:

$$f = \delta^{-1}f^{(0)}+f^{(1)}+\delta f^{(2)}+\dots$$

3.1. *CL*: $O(\delta^0)$

The functions $w^{(0)}$ and $F^{(0)}$ satisfy the equations

$$\mathcal{M}w^{(0)} \equiv Z\frac{\partial w^{(0)}}{\partial \vartheta}+\sin\vartheta\frac{\partial w^{(0)}}{\partial Z}-\lambda\frac{\partial^2 w^{(0)}}{\partial Z^2}=0; \quad \mathcal{M}F^{(0)}=0, \tag{3.9}$$

which coincide with (A 1) (see the Appendix) and may be found in almost every work in nonlinear CL. The main iteration of the solution (see (A 8)) is

$$w^{(0,0)}=\begin{cases} C\sigma\displaystyle\int_1^\kappa \frac{\mathrm{d}\zeta}{Q(\zeta)}+w_e(\tau), & \kappa>1 \\ w_e(\tau), & |\kappa|<1, \end{cases} \tag{3.10a}$$

$$F^{(0,0)}=\begin{cases} D\sigma\displaystyle\int_1^\kappa \frac{\mathrm{d}\zeta}{Q(\zeta)}+F_e(\tau), & \kappa>1 \\ F_e(\tau), & |\kappa|<1, \end{cases} \tag{3.10b}$$

where $\sigma=\operatorname{sign} Z$, $\kappa=Z^2/2+\cos\vartheta$, $Q(\kappa)=\int_0^{2\pi}\mathrm{d}\vartheta[2(\kappa-\cos\vartheta)]^{1/2}$, w_e and F_e represent the even (relative to transform (A 2)) part of the solution. The asymptotic representation (3.10) as $\kappa\to\infty$ is conveniently written immediately in terms of the variable S:

$$w^{(0,0)}\sim\delta^{-1}\frac{Cu_c'}{2\pi}(2B)^{-1/2}S+\frac{C\sigma}{8\pi}C^{(1)}+w_e+O(\delta), \tag{3.11a}$$

$$F^{(0,0)}\sim\delta^{-1}\frac{Du_c'}{2\pi}(2B)^{-1/2}S+\frac{D\sigma}{8\pi}C^{(1)}+F_e+O(\delta). \tag{3.11b}$$

Let us now match (3.11) to the solution in the ODLs.

3.2. *ODL*: $O(\delta^{-1})$

Here $\tilde{W}^{(0)}$ and $F^{(0)}$ also satisfy the same equation:

$$\frac{\partial^2\tilde{W}^{(0)}}{\partial S^2}-\frac{\partial\tilde{W}^{(0)}}{\partial\tau}=0, \quad \frac{\partial^2 F^{(0)}}{\partial S^2}-\frac{\partial F^{(0)}}{\partial\tau}=0,$$

but, according to (2.14*a*, *b*) and (3.6), they have different asymptotic representations as $S\to\infty$:

$$\tilde{W}^{(0)}\to -Squ_c'/k, \quad F^{(0)}\to 0.$$

The respective solutions $\tilde{W}^{(0)}=-Squ_c'/k$, $F^{(0)}=0$ when matched to (3.11) give (recall that $\tilde{W}=(2B)^{1/2}w$)

$$C=-2\pi q/k, \quad D=0. \tag{3.12}$$

3.3. *ODL*: $O(1)$

Functions $\tilde{W}^{(1)}(S,\tau)$ and $F^{(1)}(S,\tau)$ satisfy the equations with boundary conditions

$$\frac{\partial^2\tilde{W}^{(1)}}{\partial S^2}-\frac{\partial\tilde{W}^{(1)}}{\partial\tau}=0; \tag{3.13a}$$

$$\tilde{W}^{(1)}(\pm 0,\tau)=-(\pm qC^{(1)}/4k-w_e)(2B)^{1/2}, \quad \tilde{W}^{(1)}(\infty,\tau)=0; \tag{3.13b}$$

$$\frac{\partial^2 F^{(1)}}{\partial S^2}-\frac{\partial F^{(1)}}{\partial\tau}=-q\frac{u_c'}{ck}\frac{\partial\tilde{W}^{(1)}}{\partial\tau}, \tag{3.14a}$$

$$F^{(1)}(0,\tau)=F_e, \quad F^{(1)}(\infty,\tau)=0. \tag{3.14b}$$

The solution of (3.13) has the form

$$\tilde{W}^{(1)} = -q\frac{C^{(1)}}{8k}S\int_0^\infty d\tau'(\pi\tau'^3)^{-1/2}[2B(\tau-\tau')]^{1/2}\,e^{-S^2/4\tau'}$$
$$+\tfrac{1}{2}|S|\int_0^\infty d\tau'(\pi\tau'^3)^{-1/2}\,W_e(\tau-\tau')\,e^{-S^2/4\tau'};\quad W_e(\tau)=[2B(\tau)]^{1/2}\,w_e(\tau). \quad (3.15)$$

Its asymptotic representation as $S\to 0$,

$$\tilde{W}^{(1)} \sim -q\frac{\sigma C^{(1)}}{4k}(2B)^{1/2}+W_e(\tau)+q\frac{C^{(1)}}{4k}S\frac{d}{d\tau}\int_0^\infty\frac{d\tau'}{(\pi\tau')^{1/2}}[2B(\tau-\tau')]^{1/2}$$
$$-|S|\frac{d}{dt}\int_0^\infty\frac{d\tau'}{(\pi\tau')^{1/2}}W_e(\tau-\tau')+O(S^2), \quad (3.16)$$

contains a term proportional to $|S|$, for matching to which it will be necessary to have an even (with respect to (A 2)) perturbation in the CL of $O(\delta)$. As will be shown below, $w^{(1)}$ satisfies equation (3.9), i.e. its even part is independent of Z (see the Appendix). Consequently, matching is possible only when $w_e=0$. Similar reasoning leads to $F_e=0$. Thus:

$$w_e(\tau)=0,\quad W_e(\tau)=0;\quad F_e(\tau)=0,\quad F^{(0)}(\tau)=0, \quad (3.17)$$

i.e. $w^{(0)}$ is an odd (with respect to (A 2)) function of Z and ϑ.

The solution for $F^{(1)}$ is readily constructed from (3.15). We introduce the function $V(S,\tau)$, such that

$$\frac{\partial V}{\partial S}=\tilde{W}^{(1)};\quad \frac{\partial^2 V}{\partial S^2}-\frac{\partial V}{\partial\tau}=0,\quad V(\infty,\tau)=0.$$

Obviously,
$$V(S,\tau)=\int_{\pm\infty}^{S}dS'\,\tilde{W}^{(1)}(S',\tau),$$

where the sign of the lower limit coincides with the sign of S, and

$$F^{(1)}=-q\frac{u_c'}{2ck}S\frac{\partial V}{\partial\tau}\equiv -q^2\frac{u_c'}{8ck^2}C^{(1)}S\frac{\partial}{\partial\tau}\int_0^\infty\frac{d\tau'}{(\pi\tau')^{1/2}}[2B(\tau-\tau')]^{1/2}\,e^{-S^2/4\tau'}. \quad (3.18)$$

As $S\to 0$

$$F^{(1)}\sim -q^2\frac{u_c'}{8ck^2}C^{(1)}S\frac{d}{d\tau}\int_0^\infty\frac{d\tau'}{(\pi\tau')^{1/2}}[2B(\tau-\tau')]^{1/2}+O(S^2). \quad (3.19)$$

3.4. *CL: $O(\delta)$*

The functions $w^{(1)}$ and $F^{(1)}$ satisfy the equation $\mathcal{M}f=0$, as $w^{(0)}$ and $F^{(0)}$. It can be shown that they are also odd with respect to (A 2). In particular, the main iteration of $F^{(1)}$ has the form

$$F^{(1,0)}=\begin{cases} D_1\sigma\displaystyle\int_1^\kappa\frac{d\zeta}{Q(\zeta)}, & \kappa>1\\ 0, & |\kappa|<1\end{cases}$$

with the asymptotic representation as $|Z|\to\infty$

$$F^{(1,0)}\sim\sigma\frac{D_1}{2\pi}(|Z|+C^{(1)}/4)+O(Z^{-1}).$$

Matching to (3.19) gives

$$D_1 = -q^2 \frac{\pi}{4ck^2} C^{(1)} (2B(\tau))^{1/2} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\infty \frac{\mathrm{d}\tau'}{(\pi\tau')^{1/2}} [(2B(t-t')]^{1/2}. \tag{3.20}$$

3.5. *ODL*: $O(\delta)$

The functions $\tilde{W}^{(2)}$ and $F^{(2)}$ satisfy the same equations as do $\tilde{W}^{(1)}$ and $F^{(1)}$, but with different boundary conditions for $S = 0$. In particular,

$$\frac{\partial^2 F^{(2)}}{\partial S^2} - \frac{\partial F^{(2)}}{\partial \tau} = -q \frac{u_c'}{ck} \frac{\partial \tilde{W}^{(2)}}{\partial \tau}; \tag{3.21a}$$

$$F^{(2)}(\pm O, \tau) = \pm \frac{D_1 C^{(1)}}{8\pi}; \quad F^{(2)}(\infty, \tau) = 0. \tag{3.21b}$$

3.6. *CL contribution to the MSC*

We can now embark upon a calculation of the integrals that make a contribution to the MSC (2.15*a*, *b*)

$$J_s \equiv \int \mathrm{d}Y \langle F \sin \vartheta \rangle \quad \text{and} \quad J_c \equiv \int \mathrm{d}Y \langle F \cos \vartheta \rangle.$$

For calculating J_s, we write (2.13*b*) in terms of the variables ϑ, Y and τ:

$$ku_c' Y \frac{\partial F}{\partial \vartheta} + \frac{2k}{u_c'} B \sin \vartheta \frac{\partial F}{\partial Y} = \eta \frac{\partial^2 F}{\partial Y^2} - \mu \left[\frac{\partial F}{\partial \tau} + (\dot{\Theta} - \Omega) \frac{\partial F}{\partial \vartheta} \right] + \mu q \frac{u_c'}{ck} \left(\frac{\partial W}{\partial \tau} + \dot{\Theta} \frac{\partial W}{\partial \vartheta} \right).$$

We multiply this by Y and integrate over Y and ϑ:

$$\frac{2k}{u_c'} B \int \mathrm{d}Y\, Y \left\langle \frac{\partial F}{\partial Y} \sin \vartheta \right\rangle = \eta \int \mathrm{d}Y\, Y \left\langle \frac{\partial^2 F}{\partial Y^2} \right\rangle - \mu \int \mathrm{d}Y\, Y \left\langle \frac{\partial F}{\partial \tau} - q \frac{u_c'}{ck} \frac{\partial W}{\partial \tau} \right\rangle.$$

Upon integrating by parts in view of (2.14*a*, *b*), we obtain

$$\frac{2k}{u_c'} B \int \mathrm{d}Y \langle F \sin \vartheta \rangle = \eta \delta^2 \int \mathrm{d}Y\, Y \left\langle \frac{\partial F}{\partial \tau} - q \frac{u_c'}{ck} \frac{\partial W}{\partial \tau} \right\rangle.$$

It should be emphasized that integration is performed over the entire inner region, i.e. over the CL and over both ODLs. The CL contribution to the integral on the right is $O(1)$, while the ODLs give

$$\begin{aligned}
\int_{ODL} \mathrm{d}Y\, Y \left\langle \frac{\partial F}{\partial \tau} - q \frac{u_c'}{ck} \frac{\partial \tilde{W}}{\partial \tau} \right\rangle &= \delta^{-2} \int_{-\infty}^{\infty} \mathrm{d}S\, S \left\langle \frac{\partial F}{\mathrm{d}\tau} - q \frac{u_c'}{ck} \frac{\partial \tilde{W}}{\partial \tau} \right\rangle \\
&= \delta^{-2} \int_{-\infty}^{\infty} \mathrm{d}S\, S \left\langle \frac{\partial^2 F^{(1)}}{\partial S^2} + \delta \frac{\partial^2 F^{(2)}}{\partial S^2} + O(\delta^2) \right\rangle \\
&= \delta^{-2} \langle F^{(1)}(+0) - F^{(1)}(-0) + \delta [F^{(2)}(+0) - F^{(2)}(-0)] + O(\delta^2) \rangle.
\end{aligned}$$

In the above, $F^{(1)}(+0) = F^{(1)}(-0) = F_e = 0$, and $F^{(2)}(+0) = -F^{(2)}(-0) = D_1 C^{(1)}/8\pi$. Thus,

$$\begin{aligned}
J_s &= \eta \delta \frac{u_c'}{2kB} \frac{D_1(\tau)}{4\pi} C^{(1)} + O(\eta \delta^2) \\
&= -(\mu\eta)^{1/2} q^2 \frac{u_c'}{16ck^3} \frac{(C^{(1)})^2}{[2B(\tau)]^{1/2}} \frac{\mathrm{d}}{\mathrm{d}\tau} \int_0^\infty \frac{\mathrm{d}\tau'}{(\pi\tau')^{1/2}} [2B(\tau - \tau')]^{1/2} + O(\mu).
\end{aligned} \tag{3.22}$$

It is interesting to note that this same result can be obtained on the basis of the solution inside the CL alone (cf. Churilov & Shukhman 1987*a*). We introduce $\tilde{F} = F - D_1 Z/2\pi$ to give

$$\mathscr{M}\tilde{F} = Z\frac{\partial \tilde{F}}{\partial \vartheta} + \sin\vartheta \frac{\partial \tilde{F}}{\partial Z} - \lambda\frac{\partial^2 \tilde{F}}{\partial Z^2} = -\frac{D_1}{2\pi}\sin\vartheta.$$

We multiply this by Z and integrate over Z and ϑ:

$$\int \mathrm{d}Z\langle \tilde{F}\sin\vartheta\rangle = \lambda(\tilde{F}^+ - \tilde{F}^-), \quad \tilde{F}^{\pm} = \lim_{z\to\pm\infty}\tilde{F} = \pm\frac{D_1 C^{(1)}}{8\pi}.$$

It is easy to see that $\langle \tilde{F}\sin\vartheta\rangle = \langle F^{(1)}\sin\vartheta\rangle$; therefore,

$$\int \mathrm{d}Y\langle F^{(1)}\sin\vartheta\rangle = \frac{(2B)^{1/2}}{u_c'}\int \mathrm{d}Z\langle F^{(1)}\sin\vartheta\rangle = \frac{(2B)^{1/2}}{u_c'}\frac{\eta u_c'^2}{k(2B)^{3/2}}\frac{D_1 C^{(1)}}{4\pi},$$

whence (3.22) follows.

As far as the integral J_c is concerned, however, the contribution of the function $F^{(1)}$ to it is zero owing to the oddness of $F^{(1)}$ with respect to transform (A 2). And although the inclusion of the unsteady terms on the right-hand sides of equations (3.4) and (3.5) breaks their invariancy with respect to (A 2), it can be shown that no contribution to J_c is present up to and inluding $O(\lambda^2\delta^5) = O(\mu^{5/2}/\eta^{1/2})$, so that the 'three-dimensional' contribution to the MSC (2.15*a*) turns out to be uncompetitive compared with the 'two-dimensional' contribution (3.1). Thus, to the above accuracy it can be assumed that

$$J_c = 0. \tag{3.23}$$

3.7. *The nonlinear evolution equation in the regime of a quasi-steady nonlinear CL*

On substituting (3.2), (3.22) and (3.23) into (2.15*a, b*) and returning to the evolution variable $\xi = c\tau$, we obtain a system of NEE:

$$\frac{I_0}{k}\frac{\mathrm{d}B}{\mathrm{d}\xi} = \frac{ku_c'''}{2u_c'^4}C^{(1)}\lambda(\xi)\left(c\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} - \Omega\right)B$$
$$- q^2\frac{kC^{(1)}\lambda(\xi)B(\xi)}{4u_c'^4(\pi c\mu\eta^3)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \mathrm{d}\zeta\,\zeta^{-1/2}C^{(1)}\lambda(\xi-\zeta)B^2(\xi-\zeta). \tag{3.24}$$

$$\frac{I_0}{k}B\frac{\mathrm{d}\Theta}{\mathrm{d}\xi} = \left(\frac{\Omega}{k}I_1 + \frac{q^2}{2\mu}I_2\right)B - \frac{cku_c'''}{2u_c'^4}\frac{C^{(2)}}{\lambda(\xi)}\frac{\mathrm{d}B}{\mathrm{d}\xi}. \tag{3.25}$$

Here
$$\lambda(\xi) = \eta\frac{u_c'^2}{k}[2B(\xi)]^{-3/2}. \tag{3.26}$$

Upon eliminating $\mathrm{d}\Theta/\mathrm{d}\xi$ from this system, we get

$$\frac{\mathrm{d}B}{\mathrm{d}\xi} = \Gamma B - q^2 I_0\frac{C^{(1)}\lambda(\xi)B(\xi)}{4Du_c'^4(\pi c\mu\eta^3)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}\xi}\int_0^\infty \mathrm{d}\zeta\,\zeta^{-1/2}C^{(1)}\lambda(\xi-\zeta)B^2(\xi-\zeta), \tag{3.27}$$

where

$$\Gamma = C^{(1)}\lambda(\xi)I_2\frac{k^2u_c'''}{2Du_c'^4}(\Omega + cq^2/2k\mu) \equiv C^{(1)}\lambda(\xi)I_2\frac{k^2u_c'''}{2Du_c'^4}(\omega-\omega_{cr})/\mu \tag{3.28}$$

is a nonlinear growth rate of 'two-dimensional' perturbations in the regime of a quasi-steady nonlinear CL ($\lambda \ll 1$) (see e.g. Goldstein & Hultgren 1988), and

$$D = I_0^2/k^2 + (cku_c'''/2u_c'^4)^2 C^{(1)}C^{(2)}. \tag{3.29}$$

4. Discussion of the results

Based on the NEE (3.27) for the regime of a nonlinear CL and the NEE obtained in C&S for linear (viscous and unsteady) CL regimes we are ready to consider the evolution scenarios as a whole. For convenience, we write these equations in a brief form using the physical variables

$$A = \epsilon B, \quad x = \xi/(\mu\epsilon^{1/2}), \quad k_z^2 = \epsilon^{1/2}q^2, \quad \nu = \eta\epsilon^{3/2}:$$

viscous CL regime ($\gamma_L < \nu^{1/3}$, $A < \nu^{2/3}$)

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A - b_1\frac{k_z^2}{\nu^{3/2}}A\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}s\, s^{-1/2}[A(x-s)]^2; \tag{4.1}$$

nonlinear CL regime [$\gamma_L < \nu^{1/3}$ and $A > \nu^{2/3}$ or $\gamma_L > \nu^{1/3}$ and $A > \max(k_z^4, \gamma_L^2)$]

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \beta\gamma_L\frac{\nu}{A^{1/2}} - b_2 k_z^2\nu^{1/2}A^{-1/2}\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}s\, s^{-1/2}[A(x-s)]^{1/2}; \tag{4.2}$$

and unsteady CL regime ($\nu^{1/3} < \gamma_L < k_z^2$, $A < k_z^4$)

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \gamma_L A + b_3\,\mathrm{e}^{-\mathrm{i}\psi}k_z^2\int_0^\infty \mathrm{d}s\, s^3\int_0^1 \mathrm{d}\sigma\,\sigma^2 A(x-s)\,A(x-\sigma s)\,\bar{A}(x-(1+\sigma)s), \tag{4.3}$$

where γ_L is a (spatial) growth rate from linear theory, and the coefficients b_1, b_2, b_3, β and ψ are determined by flow and wave parameters and are $O(1)$.

At the outset we consider not too small k_z: $\nu^{1/3} \ll k_z^2 \ll 1$. By analogy with two-dimensional problems, it should be expected that the evolution of disturbances starting in the regime of a viscous CL ($\gamma_L < \nu^{1/3}$) and in the regime of an unsteady CL ($\gamma_L > \nu^{1/3}$), will differ greatly. Therefore, we will consider these cases separately.

4.1. *Transition from the regime of a viscous CL*

As has already been pointed out at the beginning of §3, the entire evolution of a disturbance that starts in the regime of a viscous CL, including the transition to the regime of a nonlinear CL, usually has a quasi-stationary character and can be represented by a unified evolution equation. To obtain it, it is necessary in (3.24) and (3.25) to replace $C^{(1)}\lambda$ and $C^{(2)}/\lambda$ with $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$, respectively (see (3.3)), with the result that the NEE (3.27), written in terms of physical variables, takes the form (cf. (4.1) and (4.2))

$$\frac{\mathrm{d}A}{\mathrm{d}x} = \left[\frac{-\Phi[\lambda(x)]}{\pi}\right]\gamma_L A - b_1\frac{k_z^2}{\nu^{3/2}}\left[\frac{-\Phi[\lambda(x)]}{\pi}\right]A(x)\frac{\mathrm{d}}{\mathrm{d}x}\int_0^\infty \mathrm{d}s\, s^{-1/2}\left[\frac{-\Phi_1[\lambda(x-s)]}{\pi}\right][A(x-s)]^2, \tag{4.4a}$$

where

$$\lambda(x) = \nu\frac{u_c'^2}{k}[2A(x)]^{-3/2}; \quad \Phi(\lambda) = \Phi_1(\lambda)\frac{D(\infty)}{D(\lambda)}; \quad b_1 = \frac{\pi^{3/2}I_0}{4c^{1/2}u_c'^4 D(\infty)}; \tag{4.4b}$$

$$\gamma_L = \frac{\pi I_2}{2D(\infty)}\frac{k^2u_c'''}{u_c'^4}(\omega_{cr}-\omega) > 0, \tag{4.4c}$$

and $$D(\lambda) = I_0^2/k^2 + (cku_c'''/2u_c'^4)^2\,\Phi_1(\lambda)\,\Phi_2(\lambda)$$

is a generalization to arbitrary values λ of the quantity $D = D(0)$ defined by (3.29). Based on (3.3) it is easy to see that the ratio $D(\lambda)/D(\infty)$ deviates only slightly from 1 as λ varies from 0 to ∞; therefore, in (4.4*a*) one can put $\Phi(\lambda) \approx \Phi_1(\lambda)$.

Equation (4.4*a*) describes not only the evolution of disturbances in the regimes of viscous and nonlinear CLs but also the transition from one regime to the other. In the limit of a viscous CL ($\lambda \gg 1$) it becomes (4.1) (cf. also equation (4.8) in C&S), and in the limit of a nonlinear CL ($\lambda \ll 1$) it becomes (4.2).

The main new feature of the problem solved compared to those considered earlier (Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988; Churilov 1989; Shukhman 1989) is that the nonlinearity threshold throughout the entire region of the viscous CL ($\gamma < \nu^{1/3}$),

$$A \sim A_1 = \gamma_L^{1/4} \nu^{3/4}/k_z \tag{4.5}$$

for not too small k_z ($k_z^2 > \nu^{1/3}$), lies below the boundary of the nonlinear CL, $A \sim \nu^{2/3}$, so that this region is reached not by an exponentially growing linear wave but by a nonlinear disturbance that evolves as

$$A \sim (\gamma_L^{1/2} \nu^{3/4}/k_z)\, x^{1/4}. \tag{4.6}$$

Therefore, the transition to regime of a nonlinear CL must be accompanied not only by the well-known reduction of growth rate (1.2) but also by a still totally unstudied transformation of the nonlinear term.

Because this nonlinear term is due to outer diffusion layers† rather than to the CL, C&S supposed that it does not change in the process of transition to a nonlinear CL regime whereas the linear growth rate γ_L is reduced according to (1.2). Guided by this hypothesis, C&S found the law of amplitude growth $A \sim x^{1/7}$.

The investigation undertaken in this paper has shown this hypothesis to be untenable. It has been found that the formation of the nonlinearity proceeds in a much more complicated manner, with the combined involvement of the CL and ODLs, and it is also reduced as much as twice (see (4.4)) rather than singly like the growth rate.

Equation (4.4*a*) can be used to construct a full evolution scenario for the originally small perturbation that starts from the regime of a viscous CL ($\gamma_L < \nu^{1/3}$). On reaching the nonlinearity threshold (4.5), its exponential growth is superseded by a quasi-stationary (i.e. the left-hand side of the NEE ($\mathrm{d}A/\mathrm{d}x$) is less than each of the terms on the right-hand side) power-law growth like (4.6). The transition to the regime of a nonlinear CL when $A \sim \nu^{2/3}$ means the replacement of NEE (4.1) with NEE (4.2). At first, the quasi-stationary character of disturbance growth is conserved, but there is a change in its law:

$$A \sim (\gamma_L^2 \nu/k_z^4)\, x. \tag{4.7}$$

We wish to note an interesting consequence of the reduction of the second ('nonlinear') term on the right-hand side of the NEE: whatever the power law of amplitude variation, the second term on the right-hand side of (4.2) decreases as $x^{-1/2}$.

With a further growth in amplitude, the terms, on the right-hand side of (4.2) decrease, while the left-hand side (growth rate) remains constant, and when

$$A \sim A_5 = k_z^8/\gamma_L^2 \tag{4.8}$$

all terms in (4.2) become of the same order of magnitude.

The coming into play of an unsteady term produces the same effect as the incorporation of an inductance coil in an electric circuit: the amplitude growth rate is

† More precisely, it is due to the interaction of 'regular' (F) and 'singular' (W) components of the perturbation which proceeds just in ODLs.

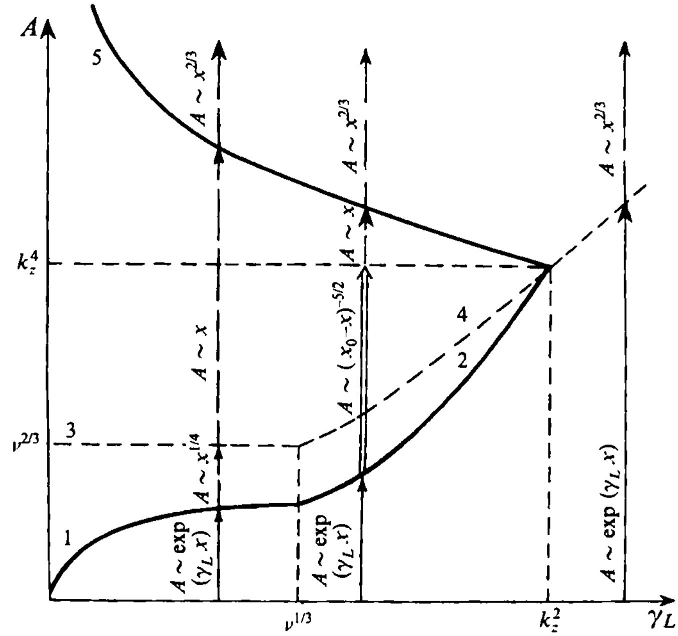

FIGURE 4. The evolution of 'weakly three-dimensional' disturbances when $k_z^2 > \nu^{1/3}$. Curve 1, $A = A_1 = \gamma_L^{1/4}\nu^{3/4}/k_z$ – nonlinearity threshold in the viscous CL regime; curve 2, $A = A_2 = \gamma_L^{5/2}/k_z$ – nonlinearity threshold in the unsteady CL regime; curves 3 and 4, formal boundaries of the nonlinear CL with the viscous and unsteady CL, respectively; curve 5, $A = A_5 = k_z^8/\gamma_L^2$ – boundary of a change of the quasi-stationary law of growth (4.7) to a 'classical' one (4.9).

decelerated, and the second term on the right-hand side of (4.2), with its invariable law of decrease, becomes uncompetitive: a 'classical' growth law in the regime of a nonlinear CL comes into play (Huerre & Scott 1980; Churilov & Shukhman 1987*a*; Goldstein & Hultgren 1988; Churilov 1989; Shukhman 1989):

$$A \sim (\gamma_L \nu)^{2/3} x^{2/3}. \tag{4.9}$$

The above evolution scenario is shown on the left-hand side of figure 4.

4.2. *Transition from the regime of an unsteady CL*

Consider now the development of a disturbance that starts from the region of an unsteady CL: $\gamma_L > \nu^{1/3}$. If the supercriticality is not too large ($\gamma_L < k_z^2$), the nonlinearity threshold

$$A \sim A_2 = \gamma_L^{5/2}/k_z \tag{4.10}$$

also lies below the formal boundary of a nonlinear CL $A \sim \gamma_L^2$. Moreover, as $A > A_2$ the amplitude growth rate is not decelerated, as in the regime of a viscous CL (see (4.6)), but is accelerated. According to (4.3), the amplitude starts to grow explosively:

$$A \sim k_z^{-1}(x_0 - x)^{-5/2}. \tag{4.11}$$

In all previous work (Churilov & Shukhman 1988; Goldstein & Leib 1989; Goldstein & Choi 1989; Shukhman 1991; Wu *et al.* 1993) the explosive stage of growth was found to be self-sustaining: an unsteady scale $l_t \sim |A^{-1}\mathrm{d}A/\mathrm{d}x|$ increased so rapidly that it always remained larger than a nonlinear scale $l_N \sim A^p$ (in our case $p = 1/2$) and the regime of the nonlinear CL could not occur in the evolution process. In our case, however, $l_t \sim (k_z A)^{2/5}$ and, because of the smallness of k_z, it becomes equal to l_N when

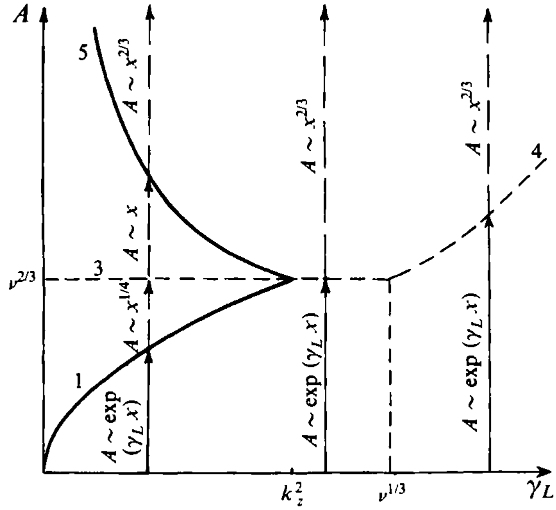

FIGURE 5. The evolution of 'weakly three-dimensional' disturbances when $k_z^2 < \nu^{1/3}$. Curves are the same as in figure 4.

$A \sim k_z^4$. This means quite a new evolution situation, namely the transition to the nonlinear CL regime from the stage of explosive growth. It should be emphasized that this is a feature of a 'weakly three-dimensional' problem: in two-dimensional problems the transition (if any) proceeds from the stage of exponential growth, while in 'strongly three-dimensional' problems in the framework of weakly nonlinear theory it does not occur at all.

The transition stage itself is a rather complicated unsteady process and requires a separate investigation. However, as has already been pointed out, the physical nature of the transient processes is the same as in the two-dimensional case where there is experience of solving similar problems. This experience shows that the relaxation must be followed by the quasi-steady nonlinear CL regime, to which the NEE (4.2) corresponds. It is easy to see that, in accordance with (4.2), the amplitude initially grows as (4.7) and then, upon reaching the level (4.8), it grows according to a 'classical' law (4.9) (figure 4, middle part).

One can see that, with increasing γ_L, the region of 'weakly three-dimensional' behaviour, i.e. growth according to (4.11) and (4.7), becomes narrow. Finally, when $\gamma_L > k_z^2$ the evolution scenario becomes totally 'two-dimensional': exponential growth up to the boundary $A \sim \gamma_L^2$ between unsteady and nonlinear CLs, followed by CL restructuring, a reduction of growth rate, and a further growth according to the law (4.9) (figure 4, right-hand side).

Thus, $\gamma_L \sim k_z^2$ serves as the right-hand boundary of the region of 'three-dimensional' behaviour on the amplitude–supercriticality diagram. If k_z is increased to $k_z = O(1)$, this boundary will be shifted to the right, the threshold of transition to the nonlinear CL regime ($A \sim k_z^4$) becomes of $O(1)$ (i.e. falls outside the validity range of weakly nonlinear theory), and we obtain a diagram for the 'strongly three-dimensional' case (C & S, figure 8). If, however, we, on the other hand, decrease k_z, then the region of 'three-dimensional' behaviour is shrinking. For $k_z^2 < \nu^{1/3}$ the respective diagram is shown in figure 5. All disturbances that start in the unsteady CL regime ($\gamma_L > \nu^{1/3}$) fall within the region of 'two-dimensional' behaviour (with a typical transition to the nonlinear CL regime from the stage of exponential growth and with a subsequent development according to the law $A \sim x^{2/3}$), so that the region of explosive growth disappears. Part of the disturbances (with $k_z^2 < \gamma_L < \nu^{1/3}$) that start from the viscous CL regime, also fall within this region. When $\gamma_L < k_z^2$ the nonlinearity threshold (4.5) is below the boundary $A \sim \nu^{2/3}$ of the nonlinear CL, and here there still remains a small region of 'three-dimensional' behaviour where, upon reaching the threshold (4.5), there is alternation of the laws of power-like growth (4.6), (4.7) and (4.9) in the process of transition to the nonlinear CL regime and of subsequent evolution in this regime.

It should be born in mind, however, that when $\gamma_L < \nu^{2/3}$ in the viscous CL regime other ('two-dimensional') nonlinearities become competitive, which will stop the growth of the disturbance and will not 'let' it pass to the nonlinear CL regime (see e.g. Churilov 1989; Shukhman 1989; Churilov & Shukhman 1992). Therefore, when $k_z^2 < \nu^{2/3}$ the three-dimensionality does not seem to manifest itself in any way.

Thus, we have shown that there is a continuous transition from a totally 'two-dimensional' to totally 'three-dimensional' evolution behaviour: with increasing k_z, the region of 'three-dimensional' behaviour on the amplitude–supercriticality diagram is ever expanding until it encompasses the entire validity range of weakly nonlinear theory ($A < 1$, $\gamma_L < 1$).

In closing we will briefly discuss the correlation between the results of our work (C & S and this paper) on the evolution of a single oblique wave, on the one hand, and results reported by Goldstein & Choi (1989) and Wu *et al.* (1993) on the evolution of a pair of oblique waves, and Wu (1993) on the evolution of a wave packet with $k_z \ll 1$, on the other. Although the cited references are concerned not only with slightly oblique waves but also with waves with $k_z \sim O(1)$, we will consider here only those results which refer to the case of slightly oblique waves.

In C & S and in the three references cited above on a pair (and on a packet) of oblique waves, evolution equations were obtained for the regime with a viscous–unsteady CL (i.e. the relationship of scales $l_t \sim l_\nu \gg l_N$).† Let these equations be written as model ones (for simplicity, we confine ourselves to the limit of an unsteady CL):

a single slightly oblique wave (SSOW)

$$\frac{\partial A}{\partial \xi} = \gamma_L A + \alpha_1 k_z^2 \frac{A^3}{\gamma^4}; \tag{4.12a}$$

a pair of slightly oblique waves (PSOW)

$$\frac{\partial A}{\partial \xi} = \gamma_L A + \alpha_2 k_z^2 \frac{A^3}{\gamma^5}. \tag{4.12b}$$

As we have already discussed in detail in C & S, the fundamental difference between these two cases is in their singularity character.

The nonlinearity thresholds, respectively, are

$$A_{thr} \sim \gamma_L^{5/2}/k_z \quad \text{for SSOW}, \tag{4.13a}$$

and

$$A_{thr} \sim \gamma_L^{3}/k_z \quad \text{for PSOW}. \tag{4.13b}$$

† To be more precise, in the paper of Goldstein & Choi (1989) the evolution equation was obtained only for the case of an unsteady CL ($l_t \gg l_\nu, l_N$), while Wu (1993*a*) deals with perturbations slightly modulated in z; however, the equation for a perturbation of the form $\sim \cos(k_z z)$ is readily reproduced from the evolution equation obtained in this paper.

The growth rates after traversing the nonlinearity threshold $A \gg A_{thr}$ are

$$\gamma \sim (Ak_z)^{2/5} \quad \text{for SSOW,} \tag{4.14a}$$

$$\gamma \sim (Ak_z)^{1/3} \quad \text{for PSOW.} \tag{4.14b}$$

As is evident, with increasing amplitude there is also an increase in growth rate (the evolution has an explosive character).

Usually, such an explosive regime in an unsteady CL is a self-maintaining one, i.e. l_t remains larger than l_N up to amplitudes $A \sim O(1)$. Such a situation occurs for the evolution of both a single oblique wave (C & S) and a pair of oblique waves (Goldstein & Choi 1989; Wu *et al.* 1993), with $k_z \sim O(1)$ as well as in the evolution problem of two-dimensional perturbations of a stratified flow (Churilov & Shukhman 1988). However the situation changes radically in the case of small k_z. In this case the scale of the nonlinear CL $l_N(\sim A^{1/2})$ can become comparable with the scale of the unsteady CL $l_t(\sim \gamma)$ for amplitudes $A \ll 1$, i.e. even in the validity range of weakly nonlinear theory.

Indeed, when

$$A \sim A_N = k_z^4 \quad \text{for SSOW} \tag{4.15a}$$

and

$$A \sim A_N = k_z^2 \quad \text{for PSOW} \tag{4.15b}$$

the CL has already ceased to be unsteady but becomes nonlinear, and hence equations (4.12*a*, *b*) become invalid. The evolution problem for perturbations in the PSOW form when $A > k_z^2$ calls for a further investigation similar to that made in the present paper, as was first mentioned by Wu (1993*a*). We emphasize that transition to the nonlinear CL regime takes place only for those perturbations that remain long wave in $z(k_z \ll 1)$ during the process of evolution. If, however, perturbation development is accompanied by a fast enough diminishing of wavelength, the CL regime may remain unsteady up to $A = O(1)$. An interesting example of such an evolution was described by Wu (1993*a*) who obtained a class of solutions in the form of waves modulated in z with simultaneous explosive growth in amplitude and 'sharpening' in z so that $k_z \sim A^{1/5}$.

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Appendix. The generalized Prandtl–Batchelor theorem, symmetry, and some properties of solutions of the equation $\mathscr{M}f = 0$

Consider the equation

$$\mathscr{M}f \equiv Z\frac{\partial f}{\partial \vartheta} + \sin\vartheta \frac{\partial f}{\partial Z} - \lambda \frac{\partial^2 f}{\partial Z^2} = 0. \tag{A 1}$$

The operator $\mathscr{M}$ is invariant under the simultaneous transformation

$$Z \to -Z, \quad \vartheta \to 2\pi - \vartheta \tag{A 2}$$

and its arbitrary solution is representable as the sum of the 'even' and the 'odd' components

$$f(Z,\vartheta) = f^+(Z,\vartheta) + f^-(Z,\vartheta), \tag{A 3}$$

$$f^+ = \tfrac{1}{2}[f(Z,\vartheta) + f(-Z, 2\pi - \vartheta)], \quad f^- = \tfrac{1}{2}[f(Z,\vartheta) - f(-Z, 2\pi-\vartheta)],$$

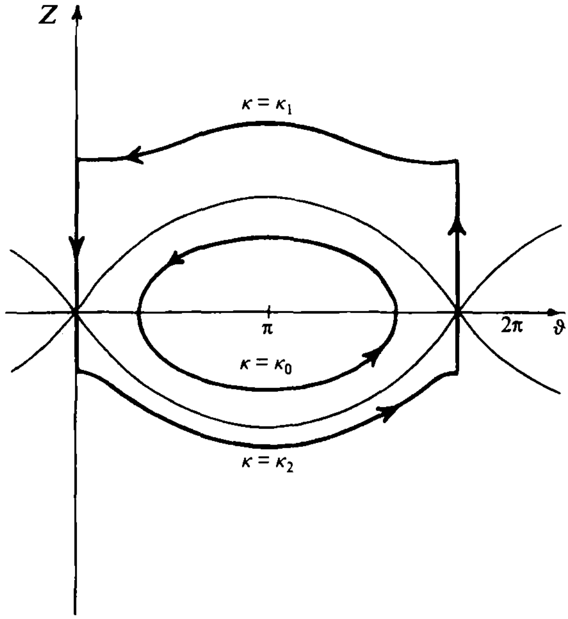

FIGURE 6. Paths of integration in the generalized Prandtl–Batchelor theorem.

each of which is a solution of (A 1). To investigate the properties of f^+ and f^-, we use the generalized Prandtl–Batchelor theorem (see, for example, Goldstein & Hultgren 1988). Equation (A 1) is equivalent to the system of equations

$$\frac{\partial^2 F}{\partial Z\,\partial \vartheta}+\sin\vartheta\frac{\partial f}{\partial Z}-\lambda\frac{\partial^2 f}{\partial Z^2}=0;\quad \frac{\partial F}{\partial Z}=Zf.$$

The first of them is integrated over Z,

$$\frac{\partial F}{\partial \vartheta}+f\sin\vartheta=\lambda\frac{\partial f}{\partial Z}+g(\vartheta),$$

and, together with the second (multiplied by $Z^{-1}\sin\vartheta$), yields

$$\left(\frac{\partial F}{\partial \vartheta}\right)_\kappa=\lambda\frac{\partial f}{\partial Z}+g(\vartheta),\quad \kappa=Z^2/2+\cos\vartheta. \tag{A 4}$$

It is customary (Goldstein & Hultgren 1988) to integrate (A 4) along a closed streamline $\kappa=\kappa_0$ (line 1 in figure 6) to give, by virtue of periodicity in ϑ,

$$\oint\frac{\partial f}{\partial Z}\mathrm{d}\vartheta=0. \tag{A 5}$$

It is easy to see, however, that (A 5) is also valid when integrated along a closed contour 2, composed of vertical segments $\vartheta=0$ and $\vartheta=2\pi$ and segments of lines $\kappa=\kappa_1$ and $\kappa=\kappa_2$, with $\kappa_1\neq\kappa_2$ in general. This is just the generalized Prandtl–Batchelor theorem for equation (A 1).

For the odd component f^-, the equality (A 5) is satisfied automatically for both contour 1 and contour 2 when $\kappa_1=\kappa_2$.

For the even component f^+, (A 5) means that

$$\int_{\pi-\vartheta_m}^{\pi+\vartheta_m}\mathrm{d}\vartheta\frac{\partial f^+}{\partial Z}=0,\quad \vartheta_m=\begin{cases}\arccos(-\kappa), & |\kappa|<1\\ \pi, & \kappa>1\end{cases} \tag{A 6}$$

Let us show that the even part of the solution of (A 1) can only be a constant (at least for $\lambda < 1$).

We seek f^+ in the form of an expansion in a power series of λ:

$$f^+ = f^{(0)} + \lambda f^{(1)} + \lambda^2 f^{(2)} + \dots .$$

It is convenient to pass to the variables κ and ϑ; equation (A 1) then takes the form

$$\left(\frac{\partial f^+}{\partial \vartheta}\right)_\kappa = \lambda \frac{\partial}{\partial \kappa}\left(Z \frac{\partial f^+}{\partial \kappa}\right). \tag{A 7}$$

The principal iteration gives $f^{(0)} = f^{(0)}(\kappa)$. Substitution into (A 6) yields

$$\int_{\pi-\vartheta_m}^{\pi+\vartheta_m} \mathrm{d}\vartheta\, Z \frac{\mathrm{d}f^{(0)}}{\mathrm{d}\kappa} = \frac{\mathrm{d}f^{(0)}}{\mathrm{d}\kappa} \int_{\pi-\vartheta_m}^{\pi+\vartheta_m} \mathrm{d}\vartheta\, Z = 0,$$

i.e. $\mathrm{d}f^{(0)}/\mathrm{d}\kappa = 0$ and

$$f^{(0)} = \begin{cases} C_{ext}, & \kappa > 1 \\ C_{in}, & |\kappa| < 1. \end{cases}$$

Because of the continuity of the solution of equation (A 1) on the boundary $\kappa = 1$ of the cat's eyes which was derived numerically by Haberman (1972) and proven mathematically by Brown & Stewartson (1978)

$$f^{(0)} = C_{in} = C_{ext} = \text{const}.$$

By subtracting this constant from f^+, the problem is brought back to the initial one. Therefore, in all iterations $f^{(n)} = \text{const}$; consequently, $f^+ = \text{const}$.

For completeness, we give here the first iteration of the odd part of the solution of f^- for $\lambda \ll 1$:

$$f^- = g^{(0)} + \lambda g^{(1)} + \dots .$$

Since f^- also satisfies equation (A 7), $g^{(0)} = g^{(0)}(\kappa)$, and from the 2π-periodicity of $g^{(0)}$ in ϑ, we find

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\left[Q_{ext}(\kappa)\frac{\mathrm{d}g^{(0)}}{\mathrm{d}\kappa}\right] = 0, \quad Q_{ext}(\kappa) = \int_0^{2\pi} \mathrm{d}\vartheta [2(\kappa - \cos\vartheta)]^{1/2}, \quad \kappa > 1,$$

$$\frac{\mathrm{d}}{\mathrm{d}\kappa}\left[Q_{in}(\kappa)\frac{\mathrm{d}g^{(0)}}{\mathrm{d}\kappa}\right] = 0, \quad Q_{in}(\kappa) = \oint_1 \mathrm{d}\vartheta [2(\kappa - \cos\vartheta)]^{1/2}, \quad |\kappa| < 1.$$

The last integral is calculated along contour 1 (figure 6). Whence it follows that

$$g^{(0)} = \begin{cases} C\sigma \displaystyle\int_1^\kappa \frac{\mathrm{d}\zeta}{Q_{ext}(\zeta)}, & \kappa > 1 \\ 0, & |\kappa| < 1, \end{cases} \tag{A 8}$$

where C is an arbitrary constant, and $\sigma = \operatorname{sign}(Z)$.

REFERENCES

BROWN, S. N. & STEWARTSON, K. 1978 The evolution of the critical layer of a Rossby wave. Part II. *Geophys. Astrophys. Fluid Dyn.* **10**, 1–24.

CHURILOV, S. M. 1989 The nonlinear stabilization of a zonal shear flow instability. *Geophys. Astrophys. Fluid Dyn.* **46**, 159–175.

CHURILOV, S. M. & SHUKHMAN, I. G. 1987*a* The nonlinear development of disturbances in a zonal shear flow. *Geophys. Astrophys. Fluid Dyn.* **38**, 145–175.

CHURILOV, S. M. & SHUKHMAN, I. G. 1987*b* Nonlinear stability of a stratified shear flow: a viscous critical layer. *J. Fluid Mech.* **180**, 1–20.

CHURILOV, S. M. & SHUKHMAN, I. G. 1988 Nonlinear stability of a stratified shear flow in the regime with an unsteady critical layer. *J. Fluid Mech.* **194**, 187–216.

CHURILOV, S. M. & SHUKHMAN, I. G. 1992 Critical layer and nonlinear evolution of disturbances in weakly supercritical shear flows *XVIIIth Intl. Congress on Theor. and Appl. Mech., Haifa, Israel, 1992.* Abstracts, pp. 39–40; Preprint ISTP No. 4–93. Irkutsk, 1993.

CHURILOV, S. M. & SHUKHMAN, I. G. 1994 Nonlinear spatial evolution of helical disturbances on an axial jet. *J. Fluid Mech.* **281**, 371–402 (referred to herein as C & S).

GOLDSTEIN, M. E. 1994 Nonlinear interactions between oblique instability waves on nearly parallel shear flows. *Phys. Fluids* **6**, 724–735.

GOLDSTEIN, M. E. & CHOI, S.-W. 1989 Nonlinear evolution of interacting oblique waves on two-dimensional shear layers. *J. Fluid Mech.* **207**, 97–120.

GOLDSTEIN, M. E. & HULTGREN, L. S. 1988 Nonlinear spatial evolution of an externally excited instability wave in a free shear layer. *J. Fluid Mech.* **197**, 295–330.

GOLDSTEIN, M. E. & LEIB, S. J. 1988 Nonlinear roll-up of externally excited free shear layers. *J. Fluid Mech.* **191**, 481–515.

GOLDSTEIN, M. E. & LEIB, S. J. 1989 Nonlinear evolution of oblique waves on compressible shear layers. *J. Fluid Mech.* **207**, 73–96.

HABERMAN, R. 1972 Critical layers in parallel flows. *Stud. Appl. Maths* **51**, 139–161.

HICKERNELL, F. J. 1984 Time-dependent critical layers in shear flows on the beta-plane. *J. Fluid Mech.* **142**, 431–449.

HUERRE, P. & SCOTT, J. F. 1980 Effects of critical layer structure on the nonlinear evolution of waves in free shear layers. *Proc. R. Soc. Lond.* A **371**, 509–524.

SHUKHMAN, I. G. 1989 Nonlinear stability of a weakly supercritical mixing layer in a rotating fluid. *J. Fluid Mech.* **200**, 425–450.

SHUKHMAN, I. G. 1991 Nonlinear evolution of spiral density waves generated by the instability of the shear layer in a rotating compressible fluid. *J. Fluid Mech.* **233**, 587–612.

SMITH, F. T. & BLENNERHASSETT, P. 1992 Nonlinear interaction of oblique three-dimensional Tollmien–Schlichting waves and longitudinal vortices. *Proc. R. Soc. Lond.* A **436**, 585–602.

STEWARTSON, K. 1978 The evolution of the critical layer of a Rossby wave. *Geophys. Astrophys. Fluid Dyn.* **9**, 185–200.

WU, X. 1993*a* Nonlinear temporal-spatial modulation of near planar Rayleigh waves in shear flows: formation of streamwise vortices. *J. Fluid Mech.* **256**, 685–719.

WU, X. 1993*b* On critical layer and diffusion layer nonlinearity in the three-dimensional stage of boundary-layer transition. *Proc. R. Soc. Lond.* A **443**, 95–106.

WU, X., LEE, S. S. & COWLEY, S. J. 1993 On the weakly nonlinear three-dimensional instability of shear layers to pairs of oblique waves: the Stokes layer as a paradigm. *J. Fluid Mech.* **253**, 681–721.